Accurate determination of the principal moments of inertia

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ABSTRACT

The moments of inertia of an object or objects are required for performing mechanical analysis. In many applications the moments of inertia are determined by measuring the distribution of the mass within a body. Given these mass distributions the inertia tensor is determined numerically, by summing the products of point masses and their distributions squared. The principal moments of inertia and the principal axes are determined from eigenvalues and eigenvectors of the inertia tensor. Here an alternative approach for determining the principal moments of inertia and the principal axes is presented; this approach is faster than the eigen-analysis approach, and is less sensitive to rounding errors.

Keywords: inertia tensor, eigenvector, eigenvalue, singular value decomposition.

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INTRODUCTION

The moments of inertia of an object or objects are required for performing mechanical analysis. For example, in human biomechanics, body segment moments of inertia are required for the computation of the resultant joint moments [e.g., 1, 2]. The analysis of the mechanical properties of bones can, in part, be quantified by assessing their moments of inertia [3]. In many applications the moments of inertia are determined by measuring the distribution of the mass within a body. This work addresses how the moments of inertia can be determined from such data while reducing the influence of errors in the measurements of the mass distribution.

In determining the moments of inertia of the body segments mass distribution of a segment has been determined using magnetic resonance imaging [e.g., 4], and dual-energy X-ray absorptiometry [e.g., 5]. In such analyses the density of the different tissues comprising the segments are assumed to be known and the mass distributions measured. Given these data the inertia tensor can be determined. It is possible to define a set of axes so the products of inertia are zero, in this case the axes are called the principal axes, and the moments of inertia (matrix diagonal terms) the principal moments of inertia.

The traditional approach for determining these principal moments of inertia and the principal axes is to compute the eigenvectors and eigenvalues of the inertia tensor [6, 7]. Here an alternative procedure will be presented which is less sensitive to errors in the measurements of the mass distribution. Therefore, the purposes of this study were to 1) present a new method determining the principal moments of inertia and their corresponding axes, and 2) to compare this new approach to the traditional approach.

METHODS

In three-dimensions for an object the inertia tensor with reference to a reference frame A is,

$$J^A = \begin{bmatrix} J_{xx} & -J_{xy} & -J_{xz} \\ -J_{yx} & J_{yy} & -J_{yz} \\ -J_{zx} & -J_{zy} & J_{zz} \end{bmatrix} \quad (1)$$

It is possible to define a set of axes so the products of inertia, non-diagonal terms, are zero. In this case the axes defining the reference frame are called the principal axes, and the moments of inertia, diagonal terms in the resulting inertia tensor, the principal moments of inertia. In the following two methods for determining the principal axes and principal moments of inertia are presented.

Method 1 – This is the approach traditionally taken based on computing the eigenvalue and eigenvector of the inertia tensor [7]. For a body made up of a certain number of particles (np), the components of the inertia tensor can be computed using the coordinates the particles (x_i, y_i, z_i) in reference system *A*, and the masses of the particles (m_i). Therefore, the components of this inertia tensor (J^A) can be computed from,

$$J^A = \begin{bmatrix} \sum^{np} m_i(y_i^2 + z_i^2) & -\sum^{np} m_i.x_i.y_i & -\sum^{np} m_i.z_i.x_i \\ -\sum^{np} m_i.x_i.y_i & \sum^{np} m_i(x_i^2 + z_i^2) & -\sum^{np} m_i.y_i.z_i \\ -\sum^{np} m_i.z_i.x_i & -\sum^{np} m_i.y_i.z_i & \sum^{np} m_i(x_i^2 + y_i^2) \end{bmatrix} \quad (2)$$

The eigenvalues of an inertia tensor are the principal moments of inertia, and the eigenvectors the principal axes [8]. The transformation of an inertia tensor to one that only contains the principal moments of inertia (J^P) is feasible because the inertia tensor is a symmetric matrix which therefore has real eigenvalues and eigenvectors, and these eigenvectors are mutually orthogonal [9]. In addition, as the tensor is positive-definitive the eigenvalues will be positive, a requirement for moments of inertia.

Method 2 – This is a new approach based on the use of the singular value decomposition of a matrix. The tensor of a body (J^A), with respect to a reference A, has the following relationship [8],

$$J^A = tr(L)\, I - L \tag{3}$$

Where,

$tr(L)$ – is the trace of matrix L

I – is the identity matrix

$L = \sum_{i=1}^{np} m_i\, p_i\ \ p_i^T$ where p_i is the x, y, and z coordinates of the point mass *i*.

All three matrices in 3 are positive-definitive symmetric matrices.

The eigenvalues of J^A are the principal moments of inertia, and are equal to the difference between the eigenvalues of $tr(L)\, I$ and L. The trace of matrix L is equal to the sum of the eigenvalues of matrix L. Therefore, if the eigenvalues of L are determined then the eigenvalues of J^A can be determined. The eigenvectors of J^A are the principal axes, which are the same as the eigenvectors of matrix L. This occurs because the eigenvectors of L are the same as those of $tr(L)\, I - L$, noting that

$(tr(L)\, I - L)\, v = tr(L)\, I\, v - L\, v$ which demonstrates an eigenvector (v) of L is also an eigenvector of $tr(L)\, I$.

The matrix L is equal to,

$$L = K\, K^T \tag{4}$$

Where,

$$K = \begin{bmatrix} x_i\sqrt{m_i} & x_{i+1}\sqrt{m_{i+1}} \ldots . & x_{np}\sqrt{m_{np}} \\ y_i\sqrt{m_i} & y_{i+1}\sqrt{m_{i+1}} \cdots & y_{np}\sqrt{m_{np}} \\ z_i\sqrt{m_i} & z_{i+1}\sqrt{m_{i+1}} \cdots & z_{np}\sqrt{m_{np}} \end{bmatrix} \tag{5}$$

For matrix J^A its eigenvalues can be determined by exploiting the properties of the singular value decomposition of K. For matrix K the singular value decomposition is determined [10],

$$K = U\, W\, V^T \tag{6}$$

Where,

U – 3 x 3 orthogonal matrix, whose columns are the eigenvectors of the matrix $K\, K^T$

W – 3 x *np* diagonal matrix comprising the three singular values ($\sigma_1, \sigma_3,\ \sigma_3$), the square of the singular values are the eigenvalues of the matrix $K\, K^T$ and $K^T K$

V – *np* x *np* orthogonal matrix, its columns are the eigenvectors of the matrix $K^T K$

Note that here the singular value decomposition relates to the eigenvectors and eigenvalues not of K but of $K^T K$ or $K^T K$. Based on equation 3 the eigenvalues of J^A can be determined from the singular values of matrix K,

$$J_{XX} = \sigma_2^2 + \sigma_3^2 \tag{7}$$

$$J_{YY} = \sigma_1^2 + \sigma_3^2 \tag{8}$$

$$J_{ZZ} = \sigma_1^2 + \sigma_2^2 \tag{9}$$

In theory the advantage of this new method for determining the inertia tensor is that it avoids the squaring of the coordinate measurements that are required in equation 2. If these measurements are numerically small, then on squaring they could test the accuracy of the floating-point representation of small numbers thus impacting the determination of the inertia tensor with respect to the principal axes. Computer computations are sensitive to calculations which test the limits of the floating-point representation of small numbers [11, 12].

EXAMPLES

Example 1: As an example of the exploitation of the new method it was applied to data previously provided [6]. In the original work the foot-pound-second (fps) system of units was used, here it is assumed these are SI units. Therefore, the system consists of four particles each of unit mass, and coordinates of $\left(0,\ \sqrt{50},\ 0\right)$, $\left(0,\ -\sqrt{50},\ 0\right)$, $(10, 0, 5)$, and $(10,\ 0, -5)$. This produces the following inertia tensor (from equation 2),

$$J^A = \begin{bmatrix} 150 & 0 & -100 \\ 0 & 250 & 0 \\ -100 & 0 & 300 \end{bmatrix} kg.m^2$$

The inertia tensor about the principal axes and rotation matrix produced by both methods were equivalent. The inertia matrix with respect to the principal axes was,

$$J^P = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 250 & 0 \\ 0 & 0 & 350 \end{bmatrix} kg.m^2$$

As would be expected for an inertia tensor $tr(J^A) = tr(J^P)$. Both methods gave equivalent principal axes, as they did for all examples. Method 1, the method based on the computation of eigenvalues and vectors, took over five times the processing time in MATLAB compared with the SVD based procedure; similar differences in processing time were determined for all examples.

Example 2: While the increased speed of the new method might make it attractive for some applications, there is another situation where it has a valuable feature. In any computer based computation, errors can occur because the numbers within the computer architecture are subject to the finite precision of the floating-point representation of a number [11].

For a system of just three particles each of unit mass, and coordinates of $\left(\frac{1}{\sqrt{2}},\ 0,\ 0\right)$, $\left(0,\ \frac{1}{\sqrt{2}},\ ,\ 0\right)$, and $\left(0,\ 0,\ \frac{1}{\sqrt{2}}\right)$ with respect to an inertial reference frame. By inspection the principal moments of inertia are $J_{XX} = J_{YY} = J_{ZZ} = 1\ kg.m^2$. As would be expected both methods give this result. If the reference frame in which these three masses are embedded, is rotated away from the inertial reference, this gives the three points masses new coordinates. If these three rotations are of 45° about the x, the y, and finally the z axes, then the inertia tensor can be recomputed using these new coordinates. The cosine and sine of 45° is an irrational number so cannot be fully represented in floating-point representations, potentially making this calculation more problematic. From the eigen-analysis the inertia tensor is,

$$J^P = \begin{bmatrix} 1 & 1.4 \times 10^{-18} & 0 \\ 1.4 \times 10^{-18} & 1 & -8.7 \times 10^{-19} \\ 0 & -8.7 \times 10^{-19} & 1 \end{bmatrix} kg.m^2$$

The small values for the products of inertia arise because of rounding errors occurring, which are exasperated by the squaring of the coordinate data in the computation of the inertia tensor described in equation 2. The SVD-analysis does not require such squared values, see equation 3, and provides the following inertia tensor,

$$J^P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} kg.m^2$$

To further illustrate the advantage of the new method a scenario is examined where one point is measured at a value close to the precision of the computer (ε). For a given computing environment to value of ε is the smallest number that a computer recognizes as being very much bigger than zero, so for example when a number below machine epsilon is added to or subtracted from a second larger number that second number does not change. For a simpler system the coordinates of the unit-point masses are now $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, and $\left(\frac{-\varepsilon}{\sqrt{2}}, \frac{-\varepsilon}{\sqrt{2}}\right)$. By inspection the eigenvalues representing the principal moments of inertia should be $1\ kg.m^2$ and $\varepsilon^2\ kg.m^2$. Letting $\varepsilon = 2.22\ \times 10^{-16}$ [13], the eigenvalues of equation 2 are $1\ kg.m^2$ and $0\ kg.m^2$, while the SVD analysis accurately produces $1\ kg.m^2$ and $\varepsilon^2\ kg.m^2$.

In the system just described one moment of inertia is much larger than the other so that for mechanical analysis the influence of the error arising from the eigen-analysis approach could be small. But in the situation where all measurements approach the limit of the floating-point representation of those measurements the eigen-analysis method could produce non-sensical results. The concept of approaching this measurement limit is compounded in the computation of the elements of equation 2, because the products of those measurements are squared.

CONCLUSION

A new method has been presented which given the locations of point masses can compute the principal moments of inertia. It has been demonstrated that this new method can estimate these moments of inertia with greater accuracy, and with greater speed than the traditional method which relies on the computation of eigenvalues and eigenvectors.

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